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THE SEDIMENTATION OF POLYDISPERSE SUSPENSIONS IN VESSELS HAVING INCLINED WALLS

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Abstract—A theory is presented for describing the sedimentation of polydisperse suspensions in two-dimensional channels having walls that are inclined to the vertical. The theory assumes that the flow is laminar and that the suspension consists of spherical beads having small particle Reynolds numbers. The suspension may consist of either N distinct species of particles or of a continuum of particle sizes and densities. For the sake of simplicity, the analysis is mostly confined to the case in which the hindered settling velocity of each particle is given by its Stokes settling velocity multiplied by a function of the total local solids concentration. Under these conditions, results are developed that are useful for the design of either batch or continuous settling devices. Experimental observations were found to be in good agreement with the predictions of the present theory.

1. INTRODUCTION

The removal of solid particles from liquid streams by means of gravity settling has application in a large variety of industrial processes and, hence, has been the subject of numerous theoretical and experimental investigations. Nearly all of these studies have been confined to monodisperse suspensions, whereas in most cases of practical inportance the suspensions contain particles having a range of settling velocities due to variations in their size, shape and density. Moreover, although it is known that the settling rate can be enhanced considerably in vessels having inclined walls—see Acrivos & Herbolzhiemer (1979)—to the authors' knowledge the effects of such polydispersity on the inclined settling process have not been considered to date. In this paper we shall investigate these effects by first confining our attention to the sedimentation of a suspension containing N distinct species of spherical particles, with the ith species being characterized by its density, ρ_i, and its radius, a_i, and shall then extend the analysis to include a continuous distribution of particle sizes and densities. We shall restrict our attention to the case where the concentration of each species is initially uniform throughout the domain (batch settling), or where the feed concentration of each species remains steady (continuous settling), and shall consider only those flows for which the Reynolds number based on the relative motion between the fluid and the particles is small.

Before proceeding with the analysis, it is instructive for us to briefly review the known results concerning batch sedimentation of bimodal suspensions in containers having vertical walls. Originally the vessel is entirely filled with suspension of uniform concentration, but, as the settling develops, the faster falling particles move away from the others thereby creating two distinct regions within the suspension. The lower region contains both species at their initial concentrations, whereas the upper region contains only particles of the slower settling species. As was shown by Smith (1966), the volume fraction of the slower species in the upper region differs from its initial value and can be calculated from the requirement of particle flux continuity across the interface separating the two regions. Smith (1966) derived the simple formula needed to evaluate this concentration jump, and extended the result to a polydisperse

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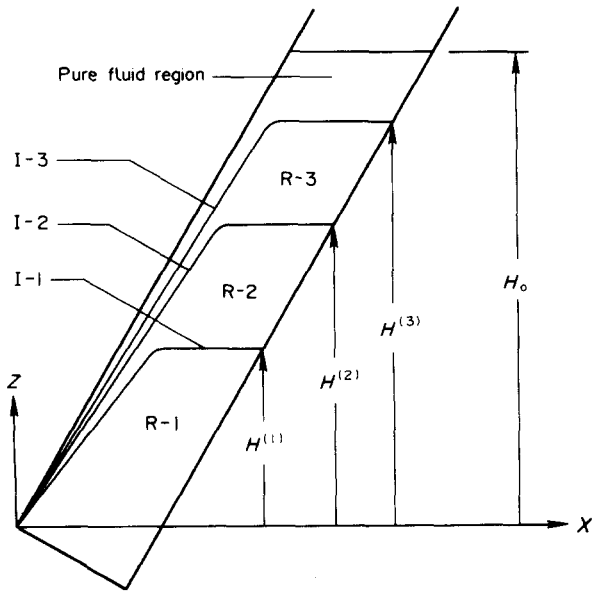

Figure 1. The different regions which develop during the sedimentation of a suspension containing three species of particles: R-1, Region 1 containing species 1, 2 and 3; R-2, Region 2 containing species 2 and 3; R-3, Region 3 containing only species 3; I-1, interface 1; I-2, interface 2; I-3, interface 3.

suspension containing N species. We shall see below that these same formulae apply when the vessel is inclined.

The major deficiency in our present understanding of vertical sedimentation of polydisperse systems is the lack of an adequate theory for predicting the settling velocity of an individual particle in the suspension. Under slow flow conditions, however, it seems reasonable to suppose that the hindered settling velocity of a given species should equal its Stokes settling velocity times a function of the local concentration of each of the species present. In fact, Lockett & Al-Habbooby (1973, 1974) obtained good agreement with experimental data for bidisperse systems by assuming that the fall velocity of a particle relative to the liquid depended only on the total local particle concentration and then evaluating this dependence by using a known correlation for monodisperse suspensions. To be sure, for systems containing large variations in the particle sizes, we would not expect this type of approach to accurately describe polydisperse settling phenomena;† however, for relatively narrow size distributions of particles, such an approximation should prove successful. In fact, as will be seen presently, the assumption that the hindered settling velocity of each particle depends only on the total local solids concentration leads to good agreement between our experiments and our theory.

It is important to bear in mind though that the behavior mentioned above—i.e. the partitioning of the suspension into distinct regions—is observed only if the total volume fraction of solids is less than some "critical" value which is typically about 0.4. Above this concentration, the various species no longer segregate readily, but due to the close packing they instead settle more or less en masse. In the present study, we shall consider only relatively low concentrations where this effect is not important.

†For example, consider a suspension consisting of small, equal-sized spheres plus a few very large spheres interspersed. Each of the small spheres would settle as if in a monodisperse suspension, whereas a large sphere would see a different environment, one more like an effective fluid. In general then, the two different-sized spheres would fall at different fractions of their Stokes settling velocities. Also, when concentrated suspensions contain particles with widely varying sizes or densities, lateral segregation of the different species may occur; however, the possible existence of such local non-uniformities shall not be considered in the present work.

2. DISCRETE POLYDISPERSE SYSTEMS

Let us examine the settling in an inclined vessel of a suspension having N distinct species of solids. As in vertical settling, N different regions are formed within the suspension (see figure 1). The lower region (Region 1) contains all of the species originally present whereas the region above it (Region 2) is devoid of the fastest settling species. In fact, each successive region contains one less type of particle so that the uppermost region (Region N) contains only particles of the slowest settling species. Of course, in addition to these N regions, concentrated sediment appears along the upward facing surface and in the bottom of the vessel; also, clear fluid accumulates at the top of the vessel and in a clear-fluid slit beneath the downward facing wall of the vessel, similar to that described by Acrivos & Herbolzheimer (1979) for monodisperse suspensions. It is our desire to calculate the concentration of each species present in each region and then to derive equations that predict the clarification rate for a given vessel geometry.

2.1 *Kinematics*

Let us begin our analysis by assuming that the suspension is random and homogeneous on a microscale and by defining an average local fluid velocity, $\bar{u}_f^*$, (where the * denotes a dimensional quantity) and an average local velocity for each type of particle, $\bar{u}_{pi}^*$, all of which vary smoothly over distances comparable to the macroscale of the system. We express $\bar{u}_f^*$ and $\bar{u}_{pi}^*$ in terms of the bulk average velocity, $\bar{u}^*$, and the average slip velocities, $\bar{u}_{si}^*$, defined as

$$\bar{u}^* = (1-c)\bar{u}_f^* + \sum_{i=1}^{N} c_i \bar{u}_{pi}^*, \qquad [2.1a]$$

and

$$\bar{u}_{si}^* = \bar{u}_{pi}^* - \bar{u}^* \quad i = 1, 2, \ldots, N, \qquad [2.1b]$$

where c_i is the local volume fraction of species i, and c is the total volume fraction of solids, i.e.

$$c = \sum_{i=1}^{N} c_i. \qquad [2.2]$$

As is evident in [2.1] we have chosen to define the slip velocities relative to the bulk average velocity rather than the average fluid velocity.

Expressions for the slip velocities simplify considerably when the particle Reynolds numbers are negligibly small and when the particle sizes are small compared to the distances over which significant variations in the bulk flow occur. With this restriction, and assuming that the effects of particle–particle interactions depend only on the total local solids concentration, we can define a function $f(c)$ so that the slip velocities are given by

$$\bar{u}_{si}^* = u_{oi}^* f(c)\bar{e} \quad i = 1, 2, \ldots, N, \qquad [2.3]$$

in which $\bar{e}$ is the unit vector in the direction of gravity, and u_{oi}^* is the Stokes settling velocity of species i,

$$u_{oi}^* = \frac{2}{9} a_i^2 \frac{(\rho_i - \rho_f)}{\mu} g,$$

where ρ_f is the density of the fluid, μ is its viscosity, and g is the gravitational constant. In general, $f(c)$ equals one for c equal zero and then decreases below unity for non-dilute systems.

Owing to the incompressibility of the particles and of fluid, the dimensionless averaged fluid and particle continuity equations can be combined to give

$$\nabla \cdot \bar{u} = 0 \qquad [2.4a]$$

and

$$\frac{\partial c_i}{\partial t} + \bar{u}_{pi} \cdot \nabla c_i = -c_i \nabla \cdot \bar{u}_{si} \qquad i = 1, 2, \ldots, N. \qquad [2.4b]$$

In the above and in what follows, all quantities have been rendered dimensionless using as the characteristic length, l, the initial vertical height, H_0^*, of the suspension (batch settling), or the vertical height, $H^{(1)*}$, of Region 1 (continuous settling), and the Stokes settling velocity, u_{01}^*, of the fastest settling particle as the characteristic velocity. Here, we have chosen $i = 1$ to denote the fastest settling species of particles in the original suspension. Finally, on substituting [2.3] into [2.4b], we obtain for the particle continuity equations

$$\frac{\partial c_i}{\partial t} + \bar{u}_{pi} \cdot \nabla c_i = -c_i u_{0i} \frac{df}{dc} \bar{e} \cdot \nabla c \qquad i = 1, 2, \ldots, N. \qquad [2.5]$$

We are now in a position to reach our first conclusion regarding the particle concentrations; specifically, that in Region 1, the concentration of each species remains at its initial value, $c_i = c_{i0}$, throughout the duration of the settling process.† This follows from [2.5] which shows that the concentration of each species must be constant along its streamlines in this region for either batch sedimentation of initially uniform suspensions or for steady-state continuous settling where feed with particle concentrations $c_i = c_{i0}$ is introduced directly into Region 1.

We now turn our attention to finding the particle concentrations in the remaining $N - 1$ regions. We number the N species of particles successively so that species j is the fastest settling particle in Region j, or

$$c_i^{(j)} = 0 \qquad i = 1, 2, \ldots, j-1, \qquad [2.6]$$

where $c_i^{(j)}$ is the local concentration of species i in Region j. Furthermore, let $Z = h^{(j)}(X, t)$ be the equation of interface j which separates Regions j and $j+1$ (see figure 1).

To determine the concentrations in Region 2, we make use of the requirement that the flux of each species of particles must be continuous at every point on interface 1. Thus, employing [2.1b] and [2.3] and the fact that $\bar{u}$ must also be continuous, we obtain

$$c_i^{(2)}[\bar{u} + u_{0i} f(c^{(2)})\bar{e} - \bar{U}] \cdot \bar{n} = c_i^{(1)}[\bar{u} + u_{0i} f(c^{(1)})\bar{e} - \bar{U}] \cdot \bar{n}$$
$$\text{at } Z = h^{(1)}(X, t) \qquad i = 1, 2, \ldots, N, \qquad [2.7]$$

where $\bar{U}$ is the velocity of the interface and $\bar{n}$ is the corresponding unit normal. Of course, as was shown above, $c_i^{(1)} = c_{i0}$ for all i throughout the duration of the settling process. Furthermore, $c_1^{(2)} \equiv 0$, so evaluating [2.7] for $i = 1$ gives

$$(\bar{u} - \bar{U}) \cdot \bar{n} + f(c^{(1)})\bar{e} \cdot \bar{n} = 0 \text{ at } Z = h^{(1)}(X, t), \qquad [2.8]$$

†This is true everywhere except near the upward-facing surface where the particle streamlines terminate and a sediment layer of high concentration forms. Although, as is well known from the theory of vertical sedimentation of monodisperse suspensions (Kynch 1952), this jump in concentration can broaden and cause concentrations to back up along particle streamlines, we would not expect this phenomena to occur for dilute suspensions, and in any case the region of higher concentration is restricted to the bottom of the vessel and to a thin layer along the upward-facing wall.

which is the kinematic conditions governing the position of interface 1. The velocity u_{01} does not explicitly appear in [2.8] because it has a dimensionless value of unity. Equations governing the bulk flow field are required in order to determine exactly the position and velocity of the interface; however, as we shall see below, they are not needed for the present purposes. Finally, substituting [2.8] into [2.7], we obtain

$$c_i^{(2)}[u_{0i}f(c^{(2)}) - f(c^{(1)})] = c_i^{(1)}[u_{0i}f(c^{(1)}) - f(c^{(1)})]$$
$$\text{at } Z = h(X,t) \quad i = 2,3,\ldots,N. \tag{2.9}$$

These $N-1$ coupled non-linear algebraic equations can be solved to calculate the concentrations in Region 2. We note that [2.9] are the same conditions as obtained by Smith (1966) for the sedimentation of discrete polydisperse suspensions in vessels with vertical walls.

An important consequence of [2.9] is that the jumps in the particle concentrations across interface 1 are independent of both position and time. Moreover, since all of the particle streamlines in Region 2 originate at interface 1, we conclude, in view of [2.5], that each $c_i^{(2)}$ must be equal to its constant value determined by [2.6] or [2.9] at every point in Region 2, independent of time. By repeating this argument for each successive region, we conclude that the concentrations are constants within any given region and that their values can be determined from [2.6] together with the conditions

$$c_i^{(j+1)}[u_{0i}f(c^{(j+1)}) - u_{0j}f(c^{(j)})] = c_i^{(j)}[u_{0i}f(c^{(j)}) - u_{0j}f(c^{(j)})]$$
$$j = 1,2,\ldots,N, \quad i = j, j+1,\ldots,N. \tag{2.10}$$

Finally, the positions of the N interfaces which separate these regions must satisfy kinematic conditions analogous to [2.8] which can be expressed as

$$\frac{\partial h^{(j)}}{\partial t} + \{\bar{u} + u_{0j}f(c^{(j)})\bar{e}\} \cdot \nabla(h^{(j)} - Z) = 0$$
$$\text{at } Z = h^{(j)}(X,t) \quad j = 1,2,\ldots,N. \tag{2.11}$$

It is worth mentioning at this point that a similar analysis would also apply if the hindered settling velocity of each species was assumed to depend on the local concentrations of all of the individual species present. Specifically, N distinct regions would again be formed within the suspension, and the concentrations in each region would be governed by [2.6] and [2.10] except that, in this case, the function f would in general be different for each of the N species and would depend on the concentrations, radii, and densities of all of the species present in a given region.

Besides completely specifying the concentration field, the results derived so far lead directly to the prediction of the settling rate for discrete polydisperse suspensions. In particular we consider settling in vessels with flat, parallel walls, and by integrating the jth equation in [2.11] along the jth interface and proceeding in a manner analogous to the monodisperse case—see Acrivos & Herbolzheimer (1979)—we obtain

$$S^{(j)}(t) = u_{0j}f(c^{(j)})\left[1 + \frac{H^{(j)}}{b}\sin\theta\right]\frac{b}{\cos\theta} \quad j = 1,2,\ldots,N, \tag{2.12}$$

where $S^{(j)}(t)$ is the dimensionless instantaneous volumetric rate at which suspension devoid of species $j, j-1,\ldots,1$ is formed—i.e. the rate at which material crosses the jth interface—θ is the angle of inclination from the vertical, $H^{(j)}(t)$ is the vertical height of the jth region (see

figure 1), and b is the spacing between the downward facing and the upward facing walls of the channel.

2.2 *Batch settling*

For batch sedimentation we are interested in determining the height of each interface and the fraction of the suspension settled at any given time. As was discussed by Acrivos & Herbolzheimer (1979) for monodisperse settling, it can be shown that the top of each of the N regions is horizontal with height $H^{(j)}(t)$ and that the remaining portions of the interface are stationary and nearly coincide with the downward-facing surface in the limit of large Λ with both R and the aspect ratio of the vessel remaining $o(\Lambda^{1/3})$,† where R is a sedimentation Reynolds number, and Λ is the ratio of a sedimentation Grashof number to R,

$$R \equiv \frac{l\rho_f u_{01}^*}{\mu}, \quad \Lambda \equiv \frac{l^2 g(\rho_1 - \rho_f)c_0}{u_{01}^* \mu}$$

Hence, in this limit, we find

$$\frac{dH^{(j)}}{dt} = -u_{0j} f(c^{(j)})\left[1 + \frac{H^{(j)}}{b}\sin\theta\right], \qquad [2.13]$$

where

$$H^{(j)} = 1 \text{ at } t = 0 \quad j = 1, 2, \ldots, N.$$

The desired predictions for batch settling are then obtained on using [2.13] in conjunction with [2.12].

2.3 *Steady-state continuous settling*

As was mentioned earlier, the concentrations in each region are governed by [2.10] for continuous systems as well as for batch systems provided that feed of uniform composition is introduced directly into Region 1. In contrast to batch systems, however, $S^{(j)}$ and $H^{(j)}$ are time-independent. In particular, the quantity of most interest is S, the volumetric rate of formation of particle-free fluid, which is given by [2.12] with $j = N$;

$$S = u_{0N} f(c^{(N)})\left[1 + \frac{H^{(N)}}{b}\sin\theta\right]\frac{b}{\cos\theta}. \qquad [2.14]$$

This equation can be used to determine the minimum size of equipment required to yield a desired settling rate, or conversely, the maximum settling rate obtainable in a given vessel.

3. CONTINUOUS PARTICLE DISTRIBUTIONS

Let us now turn our attention to the sedimentation of suspensions containing particles that have continuous distributions of sizes and densities. Since, as before, we shall assume that the slip velocities of the particles relative to the bulk flow can be expressed as their Stokes settling velocity, u_0, times a function of the total local concentration, each particle is then uniquely characterized by its Stokes settling velocity. Thus, the polydispersity can be described by the total local concentration, c, and by the local density function, $P(u_0)$, where $P(u_0)\,du_0$ is the fraction of the particles that have Stokes settling velocities between u_0 and $u_0 + du_0$, and

†Typically, in any experiment, Λ is $0(10^6)$–$0(10^8)$ and R is $0(1)$–$0(10)$ so that these conditions are easily achieved.

$cP(u_0)\,du_0$ is the local volume fraction of particles in this range. The density function is normalized;

$$\int_{U_0^{min}}^{U_0^{max}} P(u_0)\,du_0 = 1, \qquad [3.1]$$

where we have assumed that the fastest settling species has a finite Stokes settling velocity, its dimensionless value being $U_0^{max} = 1$. The slowest settling species has a Stokes settling velocity, U_0^{min}, whose value can be anywhere between zero and one; it should be kept in mind, however, that our analysis should be expected to be most accurate for narrow distributions.

As is the case with the sedimentation of discrete polydisperse suspensions, we expect that a region in the lower portion of the vessel will form where the total concentration and the density function have their initial values c_0 and $P_0(u_0)$, respectively (see figure 2). The top of this region is described by the curve $Z = h(U_0^{max}, X, t)$, above which none of the fastest settling species are present. In fact, the concentration continually decreases and only slower and slower settling particles remain as we move up the vessel. At the very top of the suspension region, described by $Z = h(U_0^{min}, X, t)$, only particles of the slowest settling species are present whose concentration, as we shall see below, may or may not be essentially zero, depending on the initial concentration and distribution function. Finally, we denote by $Z = h(U_0, X, t)$ the curve above which there are no particles of species U_0 (here, as in what follows, species U_0 refers to solid particles having a Stokes settling velocity U_0). It is our aim to find the density function and the solids concentration as functions of position and time. As we shall see presently, these are constant along any curve $Z = h(U_0, X, t)$—independent of the vessel geometry and time—with their values being determined solely by U_0, c_0 and $P_0(u_0)$.

Once again we begin by defining an average bulk velocity,

$$\bar{u} = (1-c)\bar{u}_f + c\int_{U_0^{min}}^{U_0^{max}} \bar{u}_p(u_0)P(u_0)\,du_0, \qquad [3.2]$$

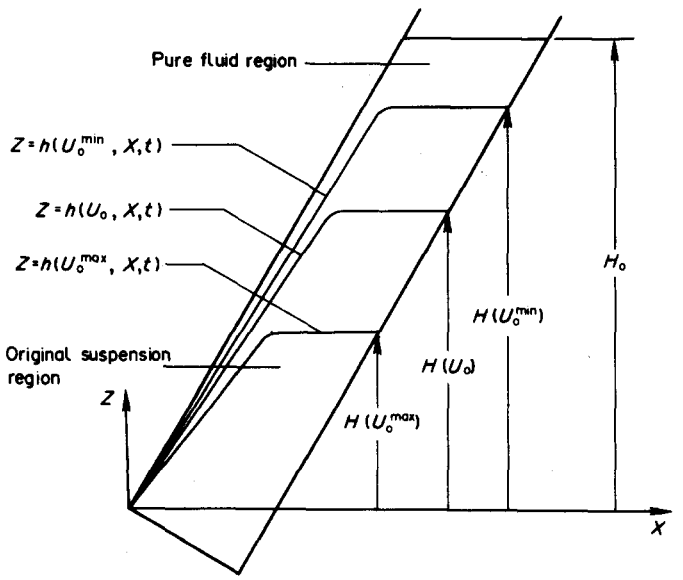

Figure 2. The different regions that develop during the sedimentation of a continuously distributed polydisperse suspension.

where $\bar{u}_p(u_0)$ is the velocity of species u_0. Also, the slip velocity of species u_0, $\bar{u}_s(u_0)$, is defined as

$$\bar{u}_s(u_0) = \bar{u}_p(u_0) - \bar{u}, \tag{3.3}$$

where, as before, we assume that

$$\bar{u}_s(u_0) = u_0 f(c)\bar{e}. \tag{3.4}$$

Using arguments similar to those already employed for the discrete case, we can show that, throughout the duration of the settling process, $c = c_0$ and $P(u_0) = P_0(u_0)$ everywhere along the curve $Z = h(U_0^{\max}, X, t)$ and in the suspension region below it provided that, for batch settling the suspension is initially uniform with $c = c_0$ and $P(u_0) = P_0(u_0)$ or that for steady-state continuous settling uniform suspension with $c = c_0$ and $P(u_0) = P_0(u_0)$ is fed directly into the region below this curve. In order to determine the concentration and the particle distribution in the upper portions of the vessel, we consider two adjacent curves, $Z = h(U_0, X, t)$ and $Z = h(U_0 + \delta U_0, X, t)$ in the vicinity of $X = X_1$ and perform a mass balance of species u_0 on the small volume element which is bounded by these two curves and by the two stationary vertical planes, $X = X_1$ and $X = X_1 + \Delta X$:

$$\begin{aligned}&[cP(u_0)(\bar{U} - \bar{u}_p(u_0)) \cdot \bar{n}A]_{U_0} - [cP(u_0)(\bar{U} - \bar{u}_p(u_0)) \cdot \bar{n}A]_{U_0+\delta U_0} \\ &+ cP(u_0)([\bar{u}_p(u_0) \cdot \bar{\imath}A]_{X_1+\Delta X} - [\bar{u}_p(u_0) \cdot \bar{\imath}A]_{X_1}) = \\ &\qquad cP(u_0)([\bar{U} \cdot \bar{n}A]_{U_0} - [\bar{U} \cdot \bar{n}A]_{U_0+\delta U_0}),\end{aligned} \tag{3.5}$$

where $\bar{U}$ is the velocity of a point on one of the two curved surfaces, $\bar{n}$ is the corresponding unit normal, $\bar{\imath}$ is the unit normal in the X-direction, and A is the area of one of the surfaces. The l.h.s. of [3.5] is the flux of species u_0 through the surfaces of the control volume whereas the r.h.s. of [3.5] is the accumulation of species u_0 within the control volume. In arriving at [3.5] we were motivated by the results of the previous section and have assumed, subject to *a posteriori* justification, that the concentration of each species is constant along any curve $Z = h(U_0, X, t)$.

We now apply the kinematic condition,

$$\bar{U} \cdot \bar{n} = \bar{u}_p(U_0) \cdot \bar{n}, \tag{3.6}$$

along with [3.3], [3.4], and the fact that the bulk flow is incompressible, to [3.5] and take the limit as $\delta U_0 \to 0$ to find,

$$\xi \frac{d(cP(u_0))}{d\xi} - u_0 \frac{d(cP(u_0)f(c))}{d\xi} = 0, \tag{3.7}$$

where $\xi \equiv U_0 f(c)$. The initial conditions are $c = c_0$ and $P(u_0) = P_0(u_0)$ when $U_0 = U_0^{\max}$; these conditions along with [3.7] are self-consistent with the above assumption that c and $P(u_0)$ are constant along the curves $Z = h(U_0, X, t)$, and these latter values can thus be determined from [3.7] together with [3.1] as a constraint. Of course, [3.7] applies only for $U_0^{\min} < u_0 \le U_0$, whereas for $U_0 < u_0 \le U_0^{\max}$ we have $P(u_0) = 0$ along $Z = h(U_0, X, t)$. Finally, the positions of these curves of constant composition must satisfy the kinematic conditions (3.6), which can be rewritten as,

$$\frac{\partial h(U_0, X, t)}{\partial t} + (\bar{u} + U_0 f(c)\bar{e}) \cdot \nabla(h(U_0, X, t) - Z) = 0. \tag{3.8}$$

In general, knowledge of the bulk flow field is needed in order to solve [3.8]; however, we can obtain some very useful information directly. For settling in vessels with straight, parallel walls, [3.8] can be integrated to give

$$S(U_0, t) = U_0 f(c)\left[1 + \frac{H(U_0, t)}{b} \sin\theta\right]\frac{b}{\cos\theta}, \qquad [3.9]$$

where $S(U_0, t)$ is the dimensionless volumetric rate at which suspension crosses the curve $Z = h(U_0, X, t)$, and $H(U_0, t)$ is the intercept of this curve with the upward-facing wall. In particular, for $U_0 = U_0^{\min}$, [3.9] is the relationship between the height of the very top of the suspension and the rate of production of clarified fluid. For steady-state, continuous operation, this is the only information required for designing a vessel that can achieve a particular clarification rate. In the limit of large Λ, with $\Lambda^{-1/3}H/b \ll 1$ and $R\Lambda^{-1/3} \ll 1$, we also expect that the top of each curve $Z = h(U_0, X, t)$ will be nearly horizontal with a height $Z = H(U_0, t)$ and that the remainder of each curve will essentially coincide with the downward facing wall. Under these conditions for batch settling, we then have

$$\frac{dH}{dt}(U_0, t) = -U_0 f(c)\left[1 + \frac{H(u_0, t)}{b} \sin\theta\right], \qquad [3.10]$$

where $H(U_0, 0) = 1$.

Finally, we note that, as was discussed for discrete polydisperse systems, the above analysis can easily be extended to include the case where the hindered settling velocity of any particle is assumed to depend on the concentration of all the individual particles locally present.

4. RESULTS, EXPERIMENTS AND DISCUSSION

Let us now consider a few illustrative examples, and let us also examine the results of several settling experiments that were performed in our laboratory in order to test the predictions of the present theory. The suspending medium was a synthetic viscous oil, and the particles were spherical glass beads. The vessel used in the experiments was 100 cm long and had a square cross-section with internal dimensions of 5 cm. This vessel could be tilted to any desired angle, and was equipped with inlet and outlet ports so that either batch or continuous settling experiments could be performed.

4.1 *Experiments with a biomodal suspension*

In order to test the predictions of Section 2, batch experiments were conducted using a suspension containing two types of particles. The slower settling particles (species 2) had a density of 2.44 g/cm^3 and were carefully sieved between standard 125 and 150 μm mesh sieves, whereas the density of species 1 was 2.99 g/cm^3, and their size range was selected by sieving between 177 and 219 μm mesh sieves. The viscosity of the fluid was 0.667 poise and its density 0.992 g/cm^3. The initial concentrations were $c_{10} = 0.03$ and $c_{20} = 0.01$ respectively.

A key assumption of the present theory is that the hindrance effect in polydisperse suspensions can be accurately predicted from monodisperse settling data. Thus, before detailed comparisons between the theoretical and experimental results could be made, vertical batch settling experiments were performed with each of the species individually. The volume fraction of solids during the experiment with species 1 was 0.04 while that during the experiment with species 2 was 0.01; the rates of the descent of the interfaces were, respectively, $v_{01}^* = 0.0427$ cm/s and $v_{02}^* = 0.0159$ cm/s, where $v_{0i}^* = u_{0i}^* f(c)$ is the vertical settling velocity of species i in a suspension with solids concentration c. These vertical settling velocities were then used to predict the concentrations in each region and the rate of fall of the interfaces that developed during the batch settling experiments performed with the bimodal system. According to our theory, the concen-

trations in Region 1 should remain at their initial values, whereas, in Region 2, $c_1^{(2)} = 0$ and $c_2^{(2)} = 0.0111$. The latter result was obtained by first noting that $c^{(2)} = c_2^{(2)}$ and then solving [2.10] iteratively for $j = 1$ and $i = 2$. As mentioned above, the vertical settling velocity of species 2, $v_{02}^*(c) = u_{02}^* f(c)$, was measured only for $c = 0.01$, whereas $c^{(1)} = 0.04$ and $c^{(2)} = 0.0111$. Corrections to the hindered settling rate of species 2 at these different local concentrations were made by applying the empirical correlation for monodisperse settling data reported by Barnea & Mizrahi (1973),

$$f(c) = \frac{(1-c)^2}{(1+c^{1/3})\exp(5c/3(1-c))}. \qquad [4.1]$$

These corrections did not require knowing the mean particle radius or the Stokes settling velocity, but were rather computed from the simple formula $v_{02}^*(c') = v_{02}^*(c) f(c')/f(c)$.

Next, a batch settling experiment was conducted with the walls of the vessel set vertically using the above suspension. The height of each interface was measured as a function of time using a cathetometer, and the data are shown in figure 3. The solid lines are the theoretical predictions for the motion of the two interfaces as determined by [2.13]; clearly our data is in good agreement with the theory. This experiment demonstrates that, at least for the present suspension, the approximation that the hindrance effect on the particle settling velocities depends only on the total concentration is accurate enough for practical purposes.

From our point of view, however, we are more concerned with the question of whether or not the present theory can be used to predict the rate of descent of the interfaces when the walls of the vessel are inclined. Using the same suspension, the heights of the two interfaces were again measured as functions of time but for angles of inclination of 20° and 40°. The data are shown in figure 4, and the solid lines are the theoretical curves obtained from [2.13]; here also, the agreement is excellent. It should be mentioned that similar experiments were also conducted with a suspension containing glass beads with equal densities but with $c_{10} = 0.025$, $a_1 \cong 66\ \mu$m and $c_{20} = 0.025$, $a_2 \cong 41\ \mu$m. Again, the experiments were in very close accord with our theory.

4.2 *Results for continuous distributions*

During the sedimentation of polydisperse suspensions containing a continuous distribution of particles, the particle concentrations are governed by [3.7]. Using several different initial distributions, we have solved this equation numerically. Our technique was to start at $U_0 = U_0^{\max}$, where the initial conditions were given, and then to decrease U_0 by small steps. At each new value of U_0, a value for c was guessed, and $P(u_0)$ was computed for several values of u_0. A standard Newton-type iteration scheme on c was then employed until [3.1] was satisfied. Shown in figure 5 are the results of such a computation for $c_0 = 0.10$ and three initial distributions. Each of these corresponded to spheres having equal densities but whose initial size distribution was a rectangle function; for curves (1), (2) and (3), the minimum particle radii were, respectively, 25%, 50% and 75% of the maximum particle radii. An interesting result is that, for the broadest distribution, the concentration just below the top of suspension, c_T, is zero, whereas, for the narrower distributions, it is not. This finite concentration at the top of the suspension is expected since it should be recalled that, for vertical settling, the velocity of the top of the suspension is $U_0^{\min} f(c_T)$ and that of the top of the region containing all species of particles is $U_0^{\max} f(c_0)$. For sufficiently narrow distributions and sufficiently large c_0, it is conceivable that the former velocity would be greater than the latter if $c_T = 0$; clearly, this cannot be the case. We mentioned also that results similar to those in figure 5 were computed for various c_0. According to these, as c_0 was increased, broader distributions yielded non-zero values of c_T and also c_T/c_0 increased for a given initial distribution.

Of greater interest to us, however, is the use of results such as those mentioned above for

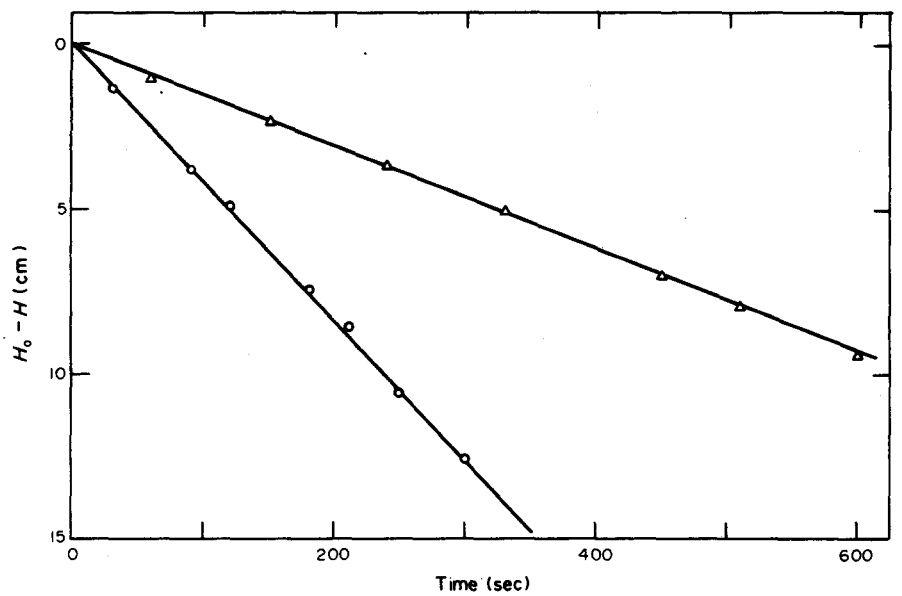

Figure 3. The height of Regions 1 and 2 vs time for the batch sedimentation of the bimodal suspension in a vessel with vertical walls: O, experimental data for Region 1, Δ, experimental data for Region 2. The solid lines are the corresponding theoretical predictions obtained from [2.13] with $\theta = 0°$.

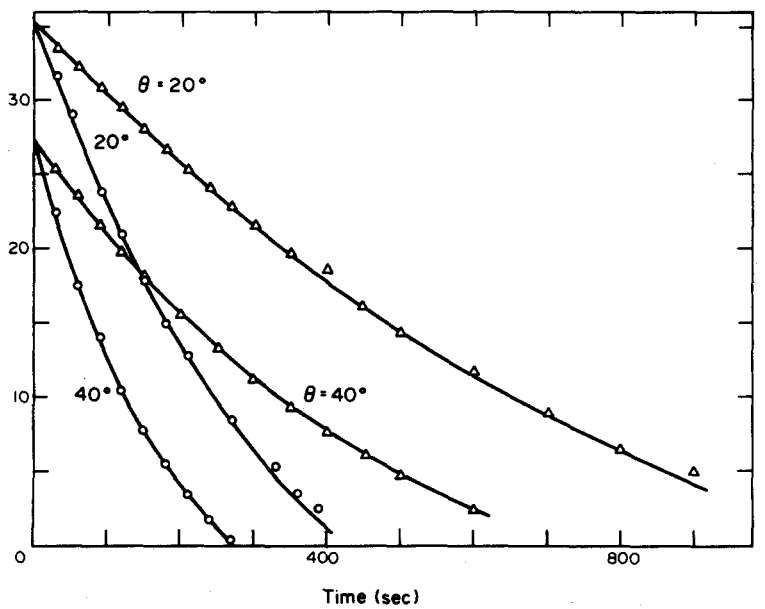

Figure 4. The height of Regions 1 and 2 vs time for the batch sedimentation of the bimodal suspension when the vessel is inclined: $b^* = 5$ cm, and $\theta = 20°$ and 40°; O, the experimental data for Region 1; Δ, experimental data for Region 2. The solid lines are the corresponding theoretical predictions obtained from [2.13].

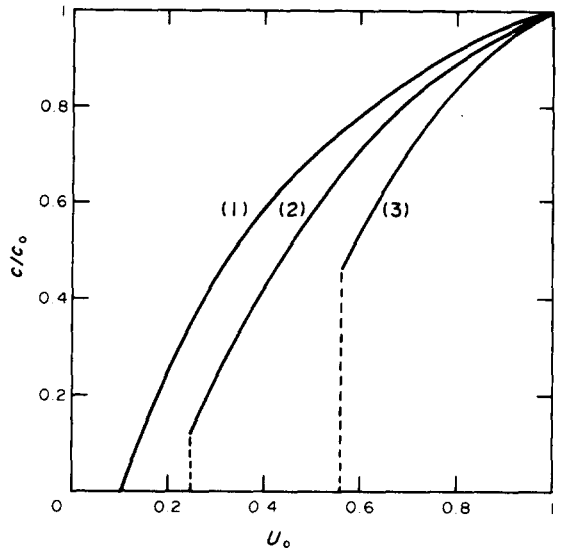

Figure 5. The total local solids concentration as determined from [3.7] and [3.1] for $c_0 = 0.10$ and three rectangle function initial distributions: (1), (2) and (3) had minimum particle radii of, respectively, 25%, 50% and 75% of the maximum particle radii.

the purpose of designing batch and continuous settling devices. For example, if the suspension corresponding to curve (2) of figure 5 were to be batch-settled in a vessel with vertical walls, the dimensionless velocity of the top of the suspension would be governed by [3.10] for $\theta = 0°$ and $U_0 = U_0^{\min}$, and would equal 0.194. In contrast, the top of a monodisperse suspension with $c_0 = 0.10$ and particles having the average radius of those of the above polydisperse suspension would fall with a velocity of 0.259. Thus, a longer time would be required to completely settle the polydisperse suspension. On the other hand, the solution of [3.7] can be combined with [3.10] to compute the time required to completely settle all of the particles of any given size. Then, if the sedimentation was terminated at this point, only spheres smaller than size would have remained unsettled, thereby suggesting a mechanism for preferentially separating the solids by size.

Let us now consider the steady-state continuous sedimentation of this same suspension in a vessel having inclined walls, which, in light of [3.9] can lead to a large enhancement in the settling rate. Let us choose, for example, $\theta = 30°$ and a clarification rate such that $b = 0.2$. Then [3.9] can be solved for $H(U_0)$;† since c is known from [3.7] as a function of U_0, we can combine these results to predict c as a function of the vertical height in the vessel. The solid line in figure 6 shows the result of this calculation, whereas the dashed line is for the sedimentation under the same circumstances of the monodisperse suspension described earlier. We see that the effect of the polydispersity is to spread out the particles in the upper portion of the settling vessel. Thus, if a completely clear overflow was required, the equipment would need to have a dimensionless height of 2.91 for the polydisperse suspension compared to only 2.09 for the monodisperse case.

So far, we have considered only the case where the overflow is located above the suspension region. If a certain tolerance of solids concentration was allowed in the overflow, a

†This calculation requires knowing $S(U_0)$. Of course, $S(U_0^{\min})$ is the rate of production of clear fluid; $S(U_0)$ differs from $S(U_0^{\min})$ by the dimensionless volumetric rate at which concentrated sediment flows down the upward facing wall at $Z = H(U_0)$. Calculation of this sediment flow rate is beyond the scope of the present work, but it is briefly discussed by Herbolzheimer (1980). For the dilute suspensions and the relatively narrow distributions considered presently, however, $S(U_0)$ can be set equal to the rate of production of clear fluid for all practical purposes.

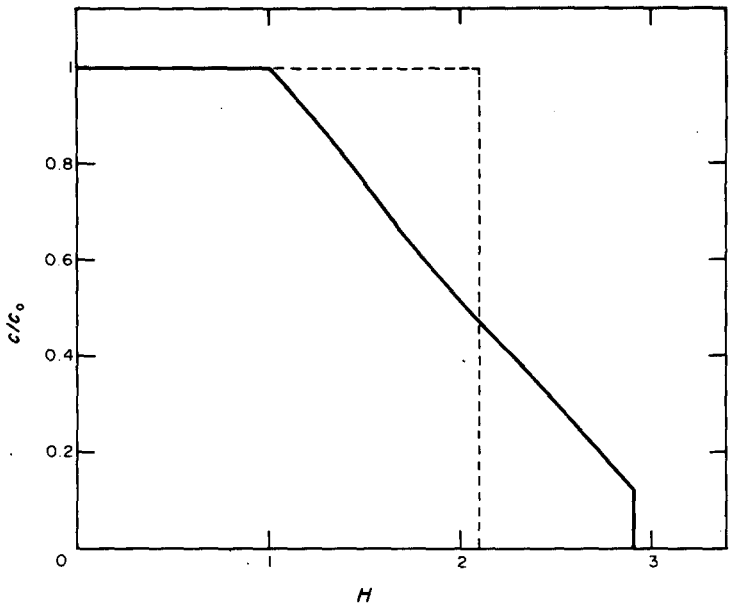

Figure 6. The total local solids concentration vs height for steady-state continuous settling with $c_0 = 0.10$, $b = 0.2$ and $= 30°$. The solid line is for distribution (2) of figure 5 as determined from [3.9], whereas the dashed line corresponds to the monodisperse case.

shorter vessel could be used, and its required height could then be calculated by performing a mass balance on the clear-fluid and the suspension entering the overflow. Finally, we point out that placing an overflow port below the top of the suspension can be used as another means of preferentially separating a certain size range of particles.

It was observed during the batch sedimentation experiments described earlier that the top of each suspension region for both the bidisperse and monodisperse experiments became somewhat fuzzy as the suspensions settled. This spreading of the interfaces appeared to be even more pronounced for steady-state continuous settling experiments, and a likely explanation is that the particles of either species were not all exactly of the same size but actually represented a continuous distribution (albeit a narrow one) of sizes.

In order to see if the observed spreading described above could be quantitatively predicted by the results of section 3, a series of steady-state continuous inclined settling experiments was performed in our laboratory using the particles of species 2 described in section 4.1. Since these beads were already sieved between the adjacent 125 and 150 μm mesh standard sieves, it was impractical to further divide the particles into several much more narrow fractions and to measure directly the monodisperse settling velocities. It was decided therefore to measure instead the particle size distribution and then to use correlations available in the literature for monodisperse suspensions in order to predict the hindered settling velocities. Sample particle sizes were determined with the aid of a microscope, and it was found that the data were well represented by a normal or "bell-shaped" size distribution function having a mean radius of 66 μ and a standard deviation of 3 μm. The density function was truncated at a maximum radius of 75 μm because all particles were sieved through a 150 μm screen. In figure 7 we have shown the results of steady-state settling experiments that had suspension with solids volume fraction of 0.02 fed in near the bottom of the vessel, a clear-fluid overflow rate of 2.57 $cm^3\ sec^{-1}$, and angles of inclination from the vertical of 20°, 30° and 40°. The suspending fluid had a viscosity of 0.60 poise and its density was 1.09 g/cm^3. The total solids concentration was determined experimentally at any location along the vessel by shining a collimated light source through the vessel and then measuring

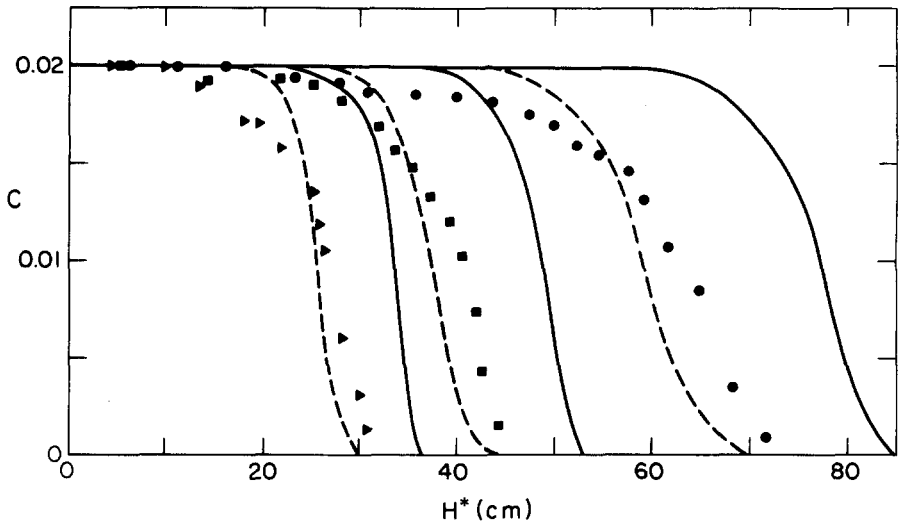

Figure 7. The total local solids concentration vs height for steady-state continuous settling of a normal distribution of particle sizes with $c_0 = 0.02$ and $S^* = 2.20\,cm^3/sec$. ●, experimental data for $\theta = 20°$; ■, experimental data for $\theta = 30°$; ▲, experimental data for $\theta = 40°$: The solid lines are the corresponding experimental predictions obtained from [3.9].

the intensity of the transmitted light. The solid lines are the theoretical predictions with $f(c)$ given by [4.1], whereas, for the dashed lines, the correlation of Richardson & Zaki (1954),

$$f(c) = (1 - c)^n, \qquad [4.2]$$

was used to determine $f(c)$, where n was chosen equal to 4.87 as suggested by Garside and Al-Dibouni (1977).

In the lower portion of the vessel, where the solids concentration is predicted to be the same as that of the feed, the measured concentrations gradually decrease to about 15% below this value. In the upper portion of the vessel, where the concentration should decrease rapidly, the experimental points fell between the theoretical curves obtained using [4.1] and [4.2]. In fact the measurements lay closer to the latter curve which is somewhat surprising since vertical settling data reported in the literature at low concentrations are better represented by [4.1] than by [4.2] (see Garside & Al-Dibouni (1977) and Barnea & Mizrahi (1973) for a discussion of the available experimental data). Similar results were also found when our experiments were repeated with $c_0 = 0.05$. One possible explanation for the observed concentration profiles of figure 7 is that the slip velocities of the particles is increased during continuous inclined settling because the resulting shear flow induces microscale structure or clumping of the particles; clearly, more data are needed to investigate this possibility.

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